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q-Analog of $A_{m-1} \oplus A_{n-1} \subset A_{mn-1}$

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A natural embedding $A_{m-1} \oplus A_{n-1} \subset A_{mn-1}$ for the corresponding quantum algebras is constructed through the appropriate comultiplication on the generators of each of the A_{m-1} and A_{n-1} algebras. The above embedding is proved in the q-boson realization by means of the isomorphism between the $\mathcal{A}_q^-(mn) \sim \otimes^n \mathcal{A}_q^-(m) \sim \otimes^m \mathcal{A}_q^-(n)$ algebras.

Recently, great interest has been given to the study of quantum algebras and their applications to physical problems. Essentially quantum algebras are Hopf algebras. A Hopf algebra is an algebra with additional structures; (i) besides the multiplication m: $A \otimes A \rightarrow A$, there is a comultiplication Δ: $A \rightarrow A \otimes A$; (ii) besides the unit 1 which provides the embedding $R \rightarrow A(C \rightarrow A)$, where $R(C)$ is the real (complex) field, there is a counit ϵ: $A \rightarrow R(C)$. All these mappings are homomorphisms and there is an antihomomorphism S: $A \rightarrow A$ called an antipode. Such algebras were developed much earlier (Sweedler, 1969; Abe, 1980) from a mathematical point of view. The contemporary development of their theory is connected with noncommutative geometry and differential calculus (Woronowicz, 1989). In physics these new mathematical objects appear in the theory of the inverse scattering problem (Faddeev *et al.*, 1988). Quantum algebras have been applied to a number of areas of physical interest, such as statistical mechanics, quantum field theory, and molecular, atomic, and nuclear physics.

In nuclear structure theory successful applications of models based on algebraic chains of Lie algebras [interaction boson model (IBM) (Iachello

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and Arima, 1987), two-vector boson model (TVBM) (Georgieva *et al.*, 1982), etc.] have been obtained.

It will be of interest to construct q-analogs of these chains and consider models based on them. The chain $su_q(3) \oplus u_q(2) \subset u_q(6)$ was already considered by Quesne (1991).

In this paper we consider the general case of the embedding

$$A^q_{m-1} \oplus A^q_{n-1} \subset A^q_{mn-1} \tag{1}$$

in the q-boson realization. The proper definition of the embedding (1) results from a careful analysis of the comultiplication structure. The present paper also provides the method of its realization, briefly described below.

As is well known, for any integer n the algebra A^q_{n-1} has a realization of its generators in terms of the q-boson algebra $\mathcal{A}_q^-(n)$ (Sun and Fu, 1989; Hayashi, 1990). In order to obtain the realization of the generators of A^q_{n-1} in terms of the q-boson algebra $\mathcal{A}_q^-(mn)$, we apply the comultiplication $m - 1$ times, then a q-boson realization for each term in the tensor product, and finally employ the isomorphism $\mathcal{A}_q^-(mn) \sim \otimes^m \mathcal{A}_q^-(n)$. By analogy we realize the generators of A^q_{m-1}. The generators of the q-deformed algebra A^q_{mn-1} have their realization by means of the same algebra $\mathcal{A}_q^-(mn)$.

We start with the algebraic relations among the regular functionals $l^{\pm}_{ij}$ of the quantum matrix group given in Faddeev *et al.* (1989):

$$\begin{aligned} \sum_{m,p} R^+_{ij,mp} l^+_{mk} l^-_{pl} &= \sum_{m,p} l^-_{jp} l^+_{im} R^+_{mp,kl} \\ \sum_{m,p} R^+_{ij,mp} l^{\pm}_{mk} l^{\pm}_{pl} &= \sum_{m,p} l^{\pm}_{jp} l^{\pm}_{im} R^+_{mp,kl} \end{aligned} \tag{2}$$

In the case of deformed A^q_{n-1} algebras the explicit form of the R^+-matrix is given by

$$R^+ = q^{1/n}\left\{ q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j=1}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j=1}^{n} e_{ij} \otimes e_{ji} \right\} \tag{3}$$

where e_{ij} are $n \times n$ matrices with elements $(e_{ij})_{km} = \delta_{ik}\delta_{jm}$.

By substituting (3) in (2) we obtain the following relations for $l^{\pm}_{ij}$:

$$\begin{aligned} [l^{(\epsilon)}_{im}, l^{(\epsilon)}_{js}] &= (1 - q)(\underbrace{l^{(\epsilon)}_{im} l^{(\epsilon)}_{js}}_{i=j} - \underbrace{l^{(\epsilon)}_{js} l^{(\epsilon)}_{im}}_{m=s}) + (q - q^{-1})(\underbrace{l^{(\epsilon)}_{jm} l^{(\epsilon)}_{is}}_{m>s} - \underbrace{l^{(\epsilon)}_{jm} l^{(\epsilon)}_{is}}_{j>i}) \\ [l^+_{im}, l^-_{js}] &= (1 - q)(\underbrace{l^+_{im} l^-_{js}}_{i=j} - \underbrace{l^-_{js} l^+_{im}}_{m=s}) + (q - q^{-1})(\underbrace{l^-_{jm} l^+_{is}}_{m>s} - \underbrace{l^+_{jm} l^-_{is}}_{j>i}) \\ \prod_{i=1}^{n} l^{\pm}_{ii} &= 1; \qquad l^+_{ii} l^-_{ii} = 1 = l^-_{ii} l^+_{ii} \\ l^+_{ij} = 0 \text{ for } i > j &\qquad \text{and} \qquad l^-_{ij} = 0 \text{ for } i < j \end{aligned} \tag{4}$$

The last relations employs not only the form of R^+, but also some additional conditions (Faddeev *et al.*, 1989).

Further, by means of the substitution

$$H_{ij} = \tilde{H}_i - \tilde{H}_j \tag{5}$$

$$l_{ij}^{\pm} = \mp q^{\pm 1/2}(q - q^{-1}) Y_{ij}^{\pm} q^{\mp(\tilde{H}_i + \tilde{H}_j)/2} \qquad \text{with} \quad Y_{ii}^{\pm} = \mp \frac{q^{\mp 1/2}}{q - q^{-1}} \tag{6}$$

one comes to the relations in Table I for the Cartan–Weyl basis of the q-deformed A_{n-1}^q algebra.

It should be noted here that the generators $Y_{ij}^{\pm}$ can be substituted by $\tilde{Y}_{ij}^{\pm} f_{ij}(q, \tilde{H})$, which will lead to modifications in the relations in Table I depending on the functions $f_{ij}(q, \tilde{H})$. An example of such a mapping from $su(2)$ to a deformed $su_q(2)$ is given in Currtright *et al.* (1991).

From the definition of the *comultiplication* $\Delta(l_{ij}^{\pm}) = \Sigma_{k=1}^{n} l_{ik}^{\pm} \otimes l_{kj}^{\pm}$ and the *counit* $\epsilon(l_{ij}^{\pm}) = \delta_{ij}$ given in Faddeev *et al.* (1988) we obtain the following coalgebraic structure:

Table I.[a]

	$[H_{ij}, H_{km}] = 0$
Borel subalgebra: $\mathcal{B}^+$	Borel subalgebra: $\mathcal{B}^-$
$[Y_{ik}^+, Y_{kj}^+]_q = Y_{ij}^+$, $i < k < j$	$[Y_{ij}^-, Y_{jk}^-]_{q^{-1}} = Y_{ik}^-$, $i > j > k$
$[Y_{ik}^+, Y_{ij}^+]_q = 0$, $i < j < k$	$[Y_{kj}^-, Y_{ij}^-]_{q^{-1}} = 0$, $i > k > j$
$[Y_{kj}^+, Y_{ij}^+]_q = 0$, $i < k < j$	$[Y_{ik}^-, Y_{ij}^-]_{q^{-1}} = 0$, $i > j > k$
$[Y_{ij}^+, Y_{km}^+] = 0$, $i < j < k < m$	$[Y_{ij}^-, Y_{km}^-] = 0$, $i > j > k > m$
$[Y_{ij}^+, Y_{km}^+] = 0$, $i < k < m < j$	$[Y_{ij}^-, Y_{km}^-] = 0$, $i > k > m > j$
$[Y_{km}^+, Y_{ij}^+] = (q - q^{-1}) Y_{kj}^i Y_{im}^+$, $i < k < j < m$	$[Y_{ij}^-, Y_{km}^-] = (q - q^{-1}) Y_{kj}^- Y_{im}^-$, $i > k > j > m$
$[H_{ik}, Y_{js}^+] = (e_i - e_k, e_j - e_s) Y_{js}^+$	$[H_{ik}, Y_{js}^-] = (e_i - e_k, e_j - e_s) Y_{js}^-$
Mixed commutators	
$[Y_{ij}^+, Y_{ji}^-] = [H_{ij}]_q$, $i < j$	
$[Y_{km}^-, Y_{ij}^+] = (q - q^{-1}) Y_{kj}^+ Y_{im}^- q^{H_{ik}}$, $j > k > i > m$	$[Y_{ij}^+, Y_{km}^-] = (q - q^{-1}) Y_{kj}^- Y_{im}^+ q^{H_{jm}}$, $k > j > m > i$
$[Y_{ij}^+, Y_{im}^-] = 0$, $j > i > m$	$[Y_{ij}^+, Y_{kj}^-] = 0$, $k > j > i$
$[Y_{ij}^+, Y_{ki}^-] = -Y_{kj}^+ q^{H_{ik}}$, $j > k > i$	$[Y_{ij}^+, Y_{ki}^-] = -q^{H_{ji}} Y_{kj}^-$, $k > j > i$
$[Y_{ij}^+, Y_{jm}^-] = Y_{im}^- q^{H_{ij}}$, $j > i > m$	$[Y_{ij}^+, Y_{jm}^-] = q^{H_{jm}} Y_{im}^+$, $j > m > i$
$[Y_{ij}^+, Y_{km}^-] = 0$	$\begin{cases} k > j > i > m;\ k > m > j > i \\ j > k > m > i;\ j > i > k > m \end{cases}$

[a]Where $(e_i, e_j) = \delta_{ij}$, the q-commutator is given by $[A, B]_q = AB - qBA$, and the q-number is defined by $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. These relations are analogous to the ones obtained by Burroughs (1990).

$$\Delta H_{ij} = H_{ij} \otimes 1 + 1 \otimes H_{ij}; \qquad \epsilon(H_{ij}) = 0; \qquad S(H_{ij}) = -H_{ij}$$

$$\epsilon(Y^{\pm}_{\overline{ij}}) = \mp \frac{q^{\mp 1/2}}{q - q^{-1}} \delta_{ij}$$

$$Y^{\pm}_{\overline{ii}} = \mp \frac{q^{\mp 1/2}}{q - q^{-1}}; \qquad Y^{+}_{\overline{ik}} = 0, \quad i > k; \qquad Y^{-}_{\overline{ik}} = 0, \quad i < k$$

$$\Delta Y^{\pm}_{\overline{ij}} = \mp (q - q^{-1}) q^{\pm 1/2} \sum_{i \le k \le j \text{ or } (j \le k \le i)} Y^{\pm}_{\overline{ik}} q^{\pm (1/2) H_{jk}} \otimes Y^{\pm}_{\overline{kj}} q^{\pm (1/2) H_{ik}} \qquad (7)$$

Applying the standard definition of the antipode S [$m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = i \circ \epsilon$], we deduce for the antipode of the generators $Y^{\mp}_{\overline{ij}}$ the following recurrent formula:

$$S(Y^{\pm}_{\overline{ij}}) = -q^{\mp 1} Y^{\pm}_{\overline{ij}} \pm (q - q^{-1}) q^{\pm 1} \sum_{i < k < j \text{ or } (i > k > j)} Y^{\pm}_{\overline{ik}} S(Y^{\pm}_{\overline{kj}}) \qquad (8)$$

Let us introduce the q-boson algebra $\mathcal{A}^{-}_{q}(n)$ with creation and annihilation operators $a^{\pm}_{i}$ and their q-boson numbers N_i as in Sun and Fu (1989), Hayashi (1990), Biedenharn (1989), and Macfarlane (1989):

$$a^{-}_{i} a^{+}_{i} - q^{\mp 1} a^{+}_{i} a^{-}_{i} = q^{\pm N_i} \qquad \text{and} \qquad [N_i, a^{\pm}_{j}] = \pm \delta_{ij} a^{\pm}_{j} \qquad (9)$$

The q-boson realization of the Cartan–Chevalley generators $H_i = H_{i,i+1}$, $Y^{+}_{i} = Y^{+}_{i,i+1}$, and $Y^{-}_{i} = Y^{-}_{i+1,i}$ of the A^{q}_{n-1} algebra given by Sun and Fu (1989) is

$$H_i = N_i - N_{i+1}; \qquad Y^{+}_{i} = a^{+}_{i} a^{-}_{i+1}; \qquad Y^{-}_{i} = a^{+}_{i+1} a^{-}_{i} \qquad (10)$$

The irreducible Fock representation $\Gamma^{[m]}_{q}$ with the vacuum state $|0\rangle$, $b^{-}_{i} |0\rangle = 0$, $N_i |0\rangle = 0$ is defined by the set of vectors

$$\Gamma^{[m]}_{q} := \left\{ |m\rangle = |m_1, \ldots, m_n\rangle = \prod_{i=1}^{n} \frac{(b^{+}_{i})^{m_i}}{([m_i]!)^{1/2}} |0\rangle \Big| m = \sum_{i=1}^{n} m_i \right\} \qquad (11)$$

with the following properties:

$$\dim \Gamma^{[m]}_{q} = \frac{(n + m - 1)!}{m!(n - 1)!}$$

$$N|m\rangle = m|m\rangle \qquad \text{where} \quad N = \sum_{i=1}^{n} N_i \qquad (12)$$

Using the definitions of H_i in (10) and N in (12), we express the operators N_i by

$$N_i = \frac{1}{n} N + \frac{1}{n} \sum_{s=2}^{n} \sum_{j=1}^{s-1} H_j - \sum_{j=1}^{i-1} H_j \qquad (13)$$

The additional generators which extend (10) to the basis of Table I of

Cartan–Weyl can be obtained from the Chevalley generators (10) by means of the first relations in the Borel subalgebras $\mathcal{B}^{\pm}$ in Table I. In this way, as in Quesne (1992), we obtain the following general realization:

$$H_{ij} = N_i - N_j; \qquad Y_{\bar{ij}}^{\pm} = a_i^+ a_j^- q^{\pm \Sigma_{i<k<j \text{ or } (j<k<i)} N_k} \tag{14}$$

Let us denote the generators of $A^q_{k_1 k_2 - 1}$ by $Y_{\bar{i}}^{\pm}$ and N_i, of $A^q_{k_1 - 1}$ by $X_{\bar{\mu}}^{\pm}$ and N_μ, of $A^q_{k_2-1}$ by $Z^{\pm s}$ and N^s, and the nth product of the comultiplication by

$$\Delta^n = \underbrace{(id \otimes id \otimes \cdots \otimes \Delta)}_{n} \underbrace{(id \otimes id \otimes \cdots \otimes \Delta)}_{n-1} \cdots (id \otimes \Delta)\Delta$$

Since Δ is a homomorphism, one can consider the following mapping:

$$A^q_{m-1} \xrightarrow{\Delta^{(n-1)}} \underbrace{A^q_{m-1} \otimes \cdots \otimes A^q_{m-1}}_{n} \tag{15}$$

For the sake of simplicity, the tensor product $\otimes$ will be dropped and the index s (or μ) will indicate the number of the tensor space. Thus we obtain

$$\begin{aligned} \tilde{H}_\mu &= \sum_{s=1}^{k_2} H^s_\mu; \qquad \tilde{X}^{\pm}_{\bar{\mu}} = \Delta^{(k_2-1)}(X^{\pm}_{\bar{\mu}}) = \sum_{s=1}^{k_2} X^{\pm s}_{\bar{\mu}} q^{\frac{1}{2}\Sigma^{k_2}_{\sigma \neq s, \sigma=1} \operatorname{sign}(\sigma - s) H^\sigma_\mu} \\ \tilde{H}^s &= \sum_{\mu=1}^{k_1} H^s_\mu; \qquad \tilde{Z}^{\pm s} = \Delta^{(k_1-1)}(Z^{\pm s}) = \sum_{\mu=1}^{k_1} Z^{\pm s}_{\bar{\mu}} q^{\frac{1}{2}\Sigma^{k_1}_{\sigma \neq \mu, \sigma=1} \operatorname{sign}(\sigma - \mu) H^s_\sigma} \end{aligned} \tag{16}$$

From the construction of the operators (16) and as a result of the used homomorphism Δ it is easy to prove that the generators $\tilde{X}^{\pm}_{\bar{\mu}}$, $\tilde{H}_\mu$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^s$ satisfy the commutations relations for the algebras $A^q_{k_1-1}$ and $A^q_{k_2-1}$.

Using the q-boson realization of the generators (14), we obtain

$$\tilde{X}^{+}_{\bar{\mu}} = \sum_{s=1}^{k_2} a^{+s}_{\mu} a^{-s}_{\mu+1} q^{\frac{1}{2}\Sigma^{k_2}_{\sigma \neq s, \sigma=1} \operatorname{sign}(\sigma - s)(N^\sigma_\mu - N^\sigma_{\mu+1})}$$

$$\tilde{X}^{-}_{\bar{\mu}} = \sum_{s=1}^{k_2} a^{+s}_{\mu+1} a^{-s}_{\mu} q^{\frac{1}{2}\Sigma^{k_2}_{\sigma \neq s, \sigma=1} \operatorname{sign}(\sigma - s)(N^\sigma_\mu - N^\sigma_{\mu+1})}$$

$$\tilde{Z}^{+s} = \sum_{\mu=1}^{k_1} a^{+s}_{\mu} a^{-s+1}_{\mu} q^{\frac{1}{2}\Sigma^{k_1}_{\sigma \neq \mu, \sigma=1} \operatorname{sign}(\sigma - \mu)(N^s_\sigma - N^{s+1}_\sigma)}$$

$$\tilde{Z}^{-s} = \sum_{\mu=1}^{k_1} a^{+s+1}_{\mu} a^{-s}_{\mu} q^{\frac{1}{2}\Sigma^{k_1}_{\sigma \neq \mu, \sigma=1} \operatorname{sign}(\sigma - \mu)(N^s_\sigma - N^{s+1}_\sigma)}$$

$$\tilde{H}^s = \sum_{\mu=1}^{k_1} N^s_u - N^{s+1}_\mu; \qquad \tilde{H}_\mu = \sum_{s=1}^{k_2} N^s_\mu - N^s_{\mu+1} \tag{17}$$

It is correct to consider the q-bosons in $\tilde{X}$ and $\tilde{Z}$ in (17) as different objects, because in $\tilde{X}$, $a_\mu^{\pm s}$ means

$$a_\mu^{\pm s} = \underbrace{id \otimes \cdots \otimes id \otimes \overbrace{a_\mu^{\pm}}^{s} \otimes id \otimes \cdots \otimes id}_{k_2}$$

while in $\tilde{Z}$

$$a_\mu^{\pm s} = \underbrace{id \otimes \cdots \otimes id \otimes \overbrace{a_s^{\pm}}^{\mu} \otimes id \otimes \cdots \otimes id}_{k_1}$$

However, in both cases, they satisfy the same relations:

$$\begin{aligned}
[a_\mu^{\pm s}, a_\nu^{\pm t}] &= 0 \qquad \text{for all} \quad s, t, \mu, \nu \\
[a_\mu^{+s}, a_\nu^{-t}] &= 0 \qquad \text{for all} \quad s \neq t;\ \mu \neq \nu \\
[N_\mu^s, a_\nu^{\pm t}] &= \pm\delta_{\mu,\nu}\delta_{s,t}a_\nu^{\pm t} \\
a_\mu^{-s}a_\mu^{+s} - q^{\mp 1}a_\mu^{+s}a_\mu^{-s} &= q^{\pm N_\mu^s}
\end{aligned} \tag{18}$$

Let us define the correspondence $i \leftrightarrow (\mu, s)$ $(k_2 \leq k_1)$:

$$i \leftrightarrow (\mu, s) \quad i = 1, \ldots, k_1k_2; \quad \mu = 1, \ldots, k_1; \quad s = 1, \ldots, k_2$$

$$\mu = 1 + \operatorname{int}\left[\frac{i-1}{k_2}\right], \qquad \text{where int}[x] \text{ is integer part of } x \tag{19}$$

$$s = 1 + (i-1)\bmod(k_2), \qquad i = (\mu - 1)k_2 + s$$

From the introduction of (19) in equations (9) and (18) it follows that the algebras $\otimes^{k_2} \mathcal{A}_q^-(k_1)$ and $\otimes^{k_1} \mathcal{A}_q^-(k_2)$ constructed by the q-bosons $a_\mu^{\pm s}$ are isomorphic to the algebra $\mathcal{A}_q^-(k_1k_2)$ constructed by the q-bosons $a_i^{\pm}$. As a result the algebras $A_{k_1-1}^q$ and $A_{k_2-1}^q$ have realizations in the $\mathcal{A}_q^-(k_1k_2)$ algebra.

Proposition 1. The generators $\tilde{X}_\mu^{\pm}$, $\tilde{H}_\mu$ commute with the generators $\tilde{Z}^{\pm s}$, $\tilde{H}^s$ given by (17).

Proof. Let us consider the commutator between the elements $\tilde{X}_\mu^+$ and $\tilde{Z}^{-s}$. For this purpose we define $Q_{t,\nu}$ and $I_{t,\nu}(\mu, s, k)$ as

$$Q_{t,\nu} = q^{\frac{1}{2}\left(\sum_{\sigma\neq t,\sigma=1}^{k_1} \operatorname{sign}(\sigma-t)(N_\mu^\sigma - N_{\mu+1}^\sigma) + \sum_{\rho\neq\nu,\rho=1}^{k_1} \operatorname{sign}(\rho-\nu)(N_\rho^s - N_\rho^{s+1})\right)}$$

$$I_{t,\nu}(\mu, s, k, q) = q^{\frac{1}{2}\sum_{\sigma\neq t,\sigma=1}^{k} \operatorname{sign}(\sigma-t)(\delta_{\mu,\nu}-\delta_{\mu+1,\nu})(\delta_{\sigma,s+1}-\delta_{\sigma,s})}$$

Using (17) and (18), we obtain for the commutator

$$[\tilde{X}^{+}_{\mu}, \tilde{Z}^{-s}] = \sum_{t=1,\nu=1}^{k_2,k_1} \{a^{+t}_{\mu} a^{t}_{\mu+1} a^{+s+1}_{\nu} a^{s}_{\nu} I_{t,\nu}(\mu, s, k_2, q) - a^{+s+1}_{\nu} a^{s}_{\nu} a^{+t}_{\mu} a^{t}_{\mu+1} I_{\nu,t}(s, \mu, k_1, q^{-1})\} Q_{t,\nu} \tag{20}$$

The sum over t and ν can be represented as a sum of five terms:

$$\begin{aligned}
(a) &= \{\nu \neq \mu, \mu + 1 \text{ and } t \neq s, s + 1\} \\
(b) &= \{\nu = \mu \text{ and } t = s + 1\} \\
(c) &= \{\nu = \mu + 1 \text{ and } t = s\} \\
(d) &= \{\nu = \mu \text{ and } t = s\} \\
(e) &= \{\nu = \mu + 1 \text{ and } t = s + 1\}
\end{aligned}$$

In these cases we have:

$$I_{t,\nu}(\mu, s, k_2, q) = \begin{cases} 1 & \text{in (a)} \\ q^{1/2} & \text{in (b), (d)} \\ q^{-1/2} & \text{in (c), (e)} \end{cases}$$

$$I_{\nu,t}(s, \mu, k_1, q^{-1}) = \begin{cases} 1 & \text{in (a)} \\ q^{1/2} & \text{in (b), (e)} \\ q^{-1/2} & \text{in (c), (d)} \end{cases}$$

In the cases (a)–(c) the bosons a^{+s+1}_{ν}, a^{s}_{ν}, a^{+t}_{μ}, and $a^{t}_{\mu+1}$ commute and the relevant terms are equal to zero. Thus the commutator is given only by the sum of (d) and (e), i.e.,

$$[\tilde{X}^{+}_{\mu}, \tilde{Z}^{-s}] = q^{-1/2} a^{+s+1}_{\mu} a^{s}_{\mu+1} (q^{-N^{s+1}_{\mu+1}} Q_{s+1,\mu+1} - q^{-N^{s}_{\mu}} Q_{s,\mu}) = 0$$

The expression $\text{sign}(\rho - \mu) = \text{sign}(\rho - \mu - 1)$ when $\rho < \mu$ or $\rho > \mu + 1$ is used essentially in the calculation of

$$q^{-N^{s+1}_{\mu+1}} Q_{s+1,\mu+1} = q^{-N^{s}_{\mu}} Q_{s,\mu}$$

The other commutators can be proved in the same way.

Further using (14) and the isomorphism (19), we have

$$a^{1+(i-1)\text{mod}(k_2)}_{1+\text{int}\{[(i-1)/k_2]\}} a^{1+(j-1)\text{mod}(k_2)}_{1+\text{int}\{[(j-1)/k_2]\}} = a^{+}_{i} a^{-}_{j} = Y^{\pm}_{ij} q^{\pm \sum_{i<\sigma<j \text{ or } (i>\sigma>j)} N_\sigma} \tag{21}$$

Finally, applying (13) and (21), we express the generators of $A^{q}_{k_1-1}$ and $A^{q}_{k_2-1}$ in (17) through the generators of $A^{q}_{k_1k_2-1}$ in the following way:

$$\tilde{Z}^{\pm s} = \sum_{\mu=1}^{k_1} Y^{\pm}_{(\mu-1)k_2+s} q^{\frac{1}{2}\sum_{\sigma\neq\mu,\sigma=1}^{k_1} \operatorname{sign}(\sigma-\mu) H_{(\sigma-1)k_2+s}}$$

$$\tilde{H}^{s} = \sum_{\mu=1}^{k_1} H_{(\mu-1)k_2+s}$$

$$\tilde{H}_{\mu} = \sum_{s=(\mu-1)k_2+1}^{(\mu-1)k_2+k_2} H_{s,s+k_2}$$

$$\tilde{X}^{+}_{\mu} = \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y^{+}_{t-k_2,t} q^{\frac{1}{2}\sum_{\nu\neq t,\nu=\mu k_2+1}^{(\mu+1)k_2} \operatorname{sign}(\nu-t) H^{\nu-k_2,\nu} + \Lambda^{+}_{t}}$$

$$\tilde{X}^{-}_{\mu} = \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y^{-}_{t,t-k_2} q^{\frac{1}{2}\sum_{\nu\neq t,\nu=\mu k_2+1}^{(\mu+1)k_2} \operatorname{sign}(\nu-t) H^{\nu-k_2,\nu} + \Lambda^{-}_{t}}$$

$$\Lambda^{\pm}_{t} = \frac{k_2-1}{k_1 k_2}\left(N + \sum_{\sigma=2}^{k_1 k_2} H_{1,\sigma}\right) \pm \sum_{\sigma=t-k_2+1}^{t-1} H_{1,\sigma} \tag{22}$$

The difference $\Lambda^{\pm}_{t}$ between the expressions for $\tilde{Z}^{\pm s}$ and $\tilde{X}^{\pm}_{\mu}$ is due to the ordering of indices in (19), which leads to the appearance of different terms

$$q^{\mp\sum_{i<k<j \text{ or } (j<k<i)} N_k}$$

in the q-boson realization (14) of the Chevalley and the additional Weyl generators. In the expression for $\Lambda^{\pm}_{t}$ the operator N in the q-boson realization has the meaning of a total number of bosons operator. In general a corresponding operator may be constructed in some extension of the algebra $A^q_{k_1k_2-1}$. This can be proved by induction. For A^q_1 $[su_q(2)]$ the operator N can be obtained from the second-order Casimir operator:

$$C^q_2 = X^-X^+ + [H/2]_q[H/2+1]_q = \frac{q^{N+1} + q^{-N-1} - q - q^{-1}}{(q-q^{-1})^2}$$

For $n > 2$, $N^{(n)}$, the corresponding operator N for A^q_{n-1}, is obtained from the recurrence

$$N^{(n)} = \frac{n+1}{n}\left\{N^{(n-1)} + \frac{1}{n+1}\sum_{t=2}^{n+1}\sum_{p=1}^{t-1} H_p - \sum_{p=1}^{n} H_p\right\} \tag{23}$$

Moreover, in practice it is only the eigenvalues of q^N which are required.

Proposition 2. The elements $\tilde{X}^{\pm}_{\mu}$, $\tilde{H}_{\mu}$ of $A^q_{k_1-1}$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^s$ of $A^q_{k_2-1}$ defined by (22) belong to the algebra $A^q_{k_1k_2-1}$ and provide an explicit embedding $A^q_{k_1-1} \oplus A^q_{k_2-1} \subset A^q_{k_1k_2-1}$ in the q-boson realization (14) of $A^q_{k_1k_2-1}$.

Proof. From the above it follows that the elements defined by (22) belong to the q-deformed $A^q_{k_1k_2-1}$ algebra. Applying the q-boson realization (14) and the correspondence (19) and (18), we obtain the q-boson realization (17) of the generators $\tilde{X}^{\pm}_{\mu}$, $\tilde{H}_{\mu}$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^s$, whose commutation relations close the algebras $A^q_{k_1-1}$ and $A^q_{k_2-1}$. Finally, these two pairs of generators commute between themselves as proved in Proposition 1, and so they close the algebra $A^q_{k_1-1} \oplus A^q_{k_2-1}$ embedded in $A^q_{k_1k_2-1}$. ∎

The results of Quesne (1991) are reproduced in the case $k_1k_2 = 6$, $k_1 = 3$, and $k_2 = 2$.

In the limit $q \to 1$ we obtain the usual embedding:

$$\tilde{H}_{\mu} = \sum_{s=(\mu-1)k_2+1}^{(\mu-1)k_2+k_2} H_{s,s+k_2}$$

$$\tilde{X}^{+}_{\mu} = \sum_{s=1}^{k_2} Y^{+}_{(\mu-1)k_2+s,\mu k_2+s}$$

$$\tilde{X}^{-}_{\mu} = \sum_{s=1}^{k_2} Y^{-}_{\mu k_2+s,(\mu-1)k_2+s}$$

$$\tilde{H}^{s} = \sum_{\mu=1}^{k_1} H_{(\mu-1)k_2+s}$$

$$\tilde{Z}^{\pm s} = \sum_{\mu=1}^{k_1} Y^{\pm}_{(\mu-1)k_2+s}$$

These results are obtained on the basis of the isomorphism between the algebras $\mathcal{A}^-_q(mn) \sim \otimes^n \mathcal{A}^-_q(m) \sim \otimes^m \mathcal{A}^-_q(n)$ and the homomorphism of the comultiplication.

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